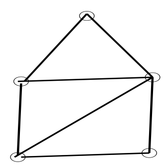
EXERCISES (HEAPS OF PIECES)

(1) Estimate the number of digits in P_{10^9}, where P_n is the n^{th} term of the Fibonacci (or Pingala) sequence, which is defined by: $P_0 := 1$, $P_1 := 1$, and $P_n := P_{n-1} + P_{n-2}$ for $n \geq 2$.

(2) Verify Stanley's theorem when G is a complete graph on n vertices (that is, every vertex is connected to every other); when G is a tree with n vertices (a tree is a connected graph with no cycles).

(3) How many acyclic orientations does the following graph have?



(4) (Project) Stanley's theorem gives an interpretation for $\gamma_G(-1)$ where $\gamma_G(\lambda)$ is the chromatic polynomial of the graph G. Extend the interpretation to $\gamma_G(-2)$, $\gamma_G(-3)$, $\ldots$.

(5) Convince yourself of the equivalence of the following two versions of the inversion lemma:
- The generating function for heaps is given by $\frac{1}{1+\mathfrak{S}}$, where $1 + \mathfrak{S}$ is the signed sum of trivial heaps.
- For any heap $\mathfrak{H}$, the sum $\sum (-1)^{\#\text{ of layers}}$ over all "layerings" of $\mathfrak{H}$ equals $(-1)^{\text{number of pieces in } \mathfrak{H}}$.

(6) (Project) Prove from first principles the second version in the inversion lemma of the previous item.

(7) Show that $\{\binom{t}{k} \mid k \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}\}$ is a $\mathbb{Z}$-basis for (the $\mathbb{Z}$-module of complex) polynomials (in one variable t) that take integral values at all integers.

(8) Prove or disprove: two trees with the same number of vertices have the same number of exact k-colourings for every k.

(9) Show that $\gamma'_G(k)$ is divisible by $k!$, where $\gamma'_G(k)$ denotes the number of exact k-colourings of a graph G.

(10) Let D_n denote the number of "Dyck paths" from the origin to (n, n): such a path consists of $2n$ steps, where each step consists of a movement of one unit to the East or one unit to the North, and the x-coordinate of any point in the path is at least as big as its y-coordinate. Let $\mathfrak{D} = \sum_{n \geq 0} D_n t^n$. Observe that $\mathfrak{D} = 1 + t\mathfrak{D}^2$ and thereby derive the following formula: $D_n = \frac{1}{n+1}\binom{2n}{n}$.